COMMENTS ON STRONGLY TORSION-FREE GROUPS

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In the present note, we discuss certain observations made by the author in February 2009 concerning *strongly torsion-free profinite groups* [cf. [Mzk2], Definition 1.1, (iii)]. These observations grew out of e-mail correspondences between the author, *Akio Tamagawa*, and *Marco Boggi*, as well as oral discussions between the author and Akio Tamagawa.

Definition 1. Let G be a *profinite group.*

(i) We shall say that G is ab-*torsion-free* if, for every open subgroup $H \subseteq G$, the abelianization H^{ab} of H is *torsion-free* [cf. Remark 1.1 below].

(ii) We shall say that G is ab-*faithful* if, for every open subgroup $H \subseteq G$, and every normal open subgroup $N \subseteq H$ of H, the natural homomorphism $H/N \to \text{Aut}(N^{\text{ab}})$ arising from conjugation is *injective.*

Remark 1.1. Note that G is *strongly torsion-free* in the sense of [Mzk2], Definition 1.1, (iii), if and only if it is topologically finitely generated and ab-*torsion-free* in the sense of Definition 1, (i).

Remark 1.2. One verifies immediately that if G is ab-faithful, then it is *slim.* Indeed, this is precisely the approach taken to verifying slimness in the proof of [Mzk2], Proposition 1.4.

Remark 1.3. It follows from Examples 3, 5 below that *neither* of the implications "ab-torsion-free $\Longrightarrow$ ab-faithful", "ab-faithful $\Longrightarrow$ ab-torsion-free" holds.

Remark 1.4. It is immediate from the definitions that every open subgroup of an ab-torsion-free (respectively, ab-faithful) profinite group is itself ab-torsion-free (respectively, ab-faithful).

Proposition 2. (Automorphisms Induced on Abelianizations) *Let G be a* **topologically finitely generated** *profinite group that satisfies at least one of the following two conditions:*

(i) G *is* ab-**faithful**.

(ii) G *is* ab-**torsion-free** *and* **pro-nilpotent**.

Let $\alpha : G \xrightarrow{\sim} G$ *be an automorphism that induces the identity automorphism on the abelianization* K^{ab} *of every characteristic open subgroup* K *of* G*. Then* α *is the identity.*

Proof. First, we recall that since G is *topologically finitely generated*, it follows that

($*^{\mathrm{char}}$) the topology of G admits a basis consisting of characteristic open subgroups.

Next, we suppose that condition (i) is satisfied. By ($*^{\mathrm{char}}$), it suffices to verify that α induces the identity automorphism on every quotient G/K, where K is a characteristic open subgroup of G. On the other hand, since [cf. (i)] the conjugation action of G/K on K^{ab} induces an injection $G/K \hookrightarrow \mathrm{Aut}(K^{\mathrm{ab}})$, the fact that α induces the identity automorphism on K^{ab} implies that α induces the identity automorphism on G/K, as desired. This completes the proof of Proposition 2 when condition (i) is satisfied.

Next, we suppose that condition (ii) is satisfied. First, let us *observe* that

($*^{\mathrm{open}}$) α induces the identity automorphism $H^{\mathrm{ab}} \xrightarrow{\sim} H^{\mathrm{ab}}$ on the abelianization of every open [i.e., not necessarily characteristic!] subgroup H of G such that $\alpha(H) = H$.

Indeed, since H^{ab} is torsion-free [cf. (ii)], this follows by observing that [cf. ($*^{\mathrm{char}}$)] there exists a characteristic open subgroup K of G such that $K \subseteq H$. That is to say, the induced map $K^{\mathrm{ab}} \to H^{\mathrm{ab}}$ has open image, hence induces a *surjection* $K^{\mathrm{ab}} \otimes \mathbb{Q} \twoheadrightarrow H^{\mathrm{ab}} \otimes \mathbb{Q}$ [i.e., where $\mathbb{Q}$ denotes the rational numbers], so the fact that α induces the identity automorphism on H^{ab} follows from the fact that α induces the identity automorphism on K^{ab}.

Now since G is *topologically finitely generated* and *pro-nilpotent* [cf. (ii)], it is well-known and easily verified that there exists an exhaustive, nilpotent sequence of characteristic open subgroups

$$\ldots \subseteq G_{n+1} \subseteq G_n \subseteq G_{n-1} \subseteq \ldots \subseteq G_0 = G$$

[i.e., $[G, G_{n-1}] \subseteq G_n$]. Suppose that m is a positive integer such that α induces the identity on G/G_{m-1}, but not on G/G_m. [Note that if there does not exist such an m, then it follows immediately that α is the identity automorphism.] Thus, there exists an element $g \in G$ such that $\alpha(g)$, g have *distinct images* in G/G_m. Then by applying ($*^{\mathrm{open}}$) to the open subgroup $H \subseteq G$ generated by g and G_{m-1}, we conclude that α induces the identity automorphism on H^{ab}. On the other hand, since H/G_m is *abelian*, this implies that $\alpha(g)$, g have the *same image* in G/G_m, a contradiction. This completes the proof of Proposition 2. $\bigcirc$

To understand the generalities discussed above, it is useful to consider the following examples. Here, the first two examples [i.e., Examples 3, 4] are constructed abstractly; the remaining examples arise from arithmetic geometry. In the following, we let Σ be a *nonempty set of prime numbers*; we shall refer to as a Σ-*integer* any positive integer each of whose prime divisors belongs to Σ.

Example 3. Semi-direct Products.

(i) Suppose that $2 \notin \Sigma$. Let M be a *nontrivial torsion-free pro-*Σ *abelian group*, which we regard as equipped with an action by $N \stackrel{\text{def}}{=} \mathbb{Z}_2$ via the morphism $N \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$ [where $\{\pm 1\}$ acts on M in the evident fashion]. Then the *semi-direct product* $G \stackrel{\text{def}}{=} M \rtimes N$ is a *nonabelian* profinite group, which is easily verified to be ab-*torsion-free*. [Indeed, if $H \subseteq G$ is an open subgroup, then one computes easily that $H^{\text{ab}} = H$ if $H \subseteq M \times (2 \cdot N)$, while $H^{\text{ab}} = H/(H \bigcap M)$ ($\hookrightarrow N$) if H is not contained in $M \times (2 \cdot N)$.] On the other hand, one verifies immediately that G *fails to be* ab-*faithful*.

(ii) The example discussed in (i) prompts the following question:

Do there exist nonabelian pro-p [where p is a prime number] groups which are ab-*torsion-free, but not* ab-*faithful?*

The author does not know the answer to this question at the time of writing.

Example 4. Graphs of Anabelioids.

(i) Let $\mathcal{G}$ be a *graph of anabelioids* [cf. [Mzk1], Definition 2.1] whose underlying graph $\mathbb{G}$ is *finite* and *connected*. Suppose further that the anabelioid $\mathcal{G}_e$ at each edge e of $\mathbb{G}$ is the anabelioid associated to the trivial group, while the anabelioid $\mathcal{G}_v$ at each edge v of $\mathbb{G}$ is the anabelioid associated to a *pro-*Σ *surface group* G_v [cf. Example 6, (i), below]. Write G for the *maximal pro-*Σ *quotient* of the fundamental group of $\mathcal{G}$ [cf. the discussion following [Mzk1], Definition 2.1]. Since $\mathbb{G}$ is finite, it follows immediately that G is *topologically finitely generated*.

(ii) One verifies immediately that the abelianization G^{ab} fits into a natural exact sequence

$$1 \to \bigoplus_v G_v^{\text{ab}} \to G^{\text{ab}} \to \pi_1(\mathbb{G})^{\text{ab}} \otimes \widehat{\mathbb{Z}}^{\Sigma} \to 1$$

— where the direct sum ranges over the vertices v of $\mathbb{G}$; $\pi_1(\mathbb{G})$ is the usual topological fundamental group of the graph $\mathbb{G}$; $\widehat{\mathbb{Z}}^{\Sigma}$ is the pro-Σ completion of $\mathbb{Z}$. In particular, it follows immediately from the fact that each G_v^{ab} is torsion-free [cf.

Example 6, (i), below], together with the well-known fact that $\pi_1(\mathbb{G})$ is a free discrete group, that G^{ab} is torsion-free. Since any connected finite étale covering of $\mathcal{G}$ is easily verified to be a graph of anabelioids as in (i), it follows by applying the above exact sequence to such connected finite étale coverings of $\mathcal{G}$ that G is ab-*torsion-free*.

(iii) Next, let us observe that G is ab-*faithful*. Indeed, it suffices to verify that if $\mathcal{G}'' \to \mathcal{G}'$ is any cyclic finite étale covering of degree l, for $l \in \Sigma$, of graphs of anabelioids as in (i), then $\Gamma = \mathrm{Gal}(\mathcal{G}''/\mathcal{G}')$ acts *nontrivially* on the abelianization $(G'')^{\mathrm{ab}}$ of the maximal pro-Σ quotient G'' of the fundamental group of $\mathcal{G}''$. To this end, observe that if Γ acts *freely* on $\mathbb{G}''$ [where we use primed and double-primed notation for the objects to associated to $\mathcal{G}''$, $\mathcal{G}'$ which are analogous to the objects associated to $\mathcal{G}$ in (i)], then the nontriviality of the action of Γ on $(G'')^{\mathrm{ab}}$ follows from the ab-faithfulness of a free pro-Σ group of finite rank [cf. Example 6, (i), below]. Thus, it suffices to consider the case where Γ $(\cong \mathbb{Z}/l\mathbb{Z})$ *fixes* some vertex v'' of $\mathbb{G}''$ lying over a vertex v' of $\mathbb{G}'$, hence arises as a quotient $G'_{v'} \twoheadrightarrow \Gamma$ [whose kernel may be identified with $G''_{v''}$]. But then the nontriviality of the action of Γ on $(G'')^{\mathrm{ab}}$ follows from the ab-faithfulness of the pro-Σ surface group $G'_{v'}$ [cf. Example 6, (i), below], together with the exact sequence of (ii), applied to $\mathcal{G}''$. This completes the proof that G is ab-faithful.

(iv) Finally, let us observe that if there exist two distinct vertices v, w of $\mathbb{G}$ such that the surface groups G_v, G_w arise from *proper hyperbolic curves* [so $H^2(G_v, \mathbb{F}_l) \cong H^2(G_w, \mathbb{F}_l) \cong \mathbb{F}_l$, for $l \in \Sigma$], then it follows that $\dim_{\mathbb{F}_l}(H^2(G, \mathbb{F}_l)) \geq 2$ — a fact that implies that, in this case,

G fails to be isomorphic to a pro-Σ surface group.

Indeed, it follows from the definitions that we have well-defined injective outer homomorphisms $\iota_v : G_v \hookrightarrow G$, $\iota_w : G_w \hookrightarrow G$. Moreover, by thinking of G as an inductive limit in the category of pro-Σ groups and considering the system of homomorphisms $G_u \to G_v$ (respectively, $G_u \to G_w$), where u ranges over the vertices of $\mathbb{G}$, given by the identity when $u = v$ (respectively, $u = w$) and the trivial homomorphism when $u \neq v$ (respectively, $u \neq w$), one obtains a surjection $\rho_v : G \twoheadrightarrow G_v$ (respectively, $\rho_w : G \twoheadrightarrow G_w$) such that the outer homomorphism $\rho_v \circ \iota_v$ (respectively, $\rho_w \circ \iota_w$) is the identity on G_v (respectively, G_w), while the outer homomorphism $\rho_v \circ \iota_w$ (respectively, $\rho_w \circ \iota_v$) is the trivial homomorphism on G_w (respectively, G_v). Thus, the pairs (ρ_v, ρ_w) and (ι_v, ι_w) induce morphisms

$$H^2(G_v, \mathbb{F}_l) \times H^2(G_w, \mathbb{F}_l) \to H^2(G, \mathbb{F}_l) \to H^2(G_v, \mathbb{F}_l) \times H^2(G_w, \mathbb{F}_l)$$

whose composite is the *identity*. But this implies that $\dim_{\mathbb{F}_l}(H^2(G, \mathbb{F}_l)) \geq 2$, as desired.

Example 5. Local Absolute Galois Groups. Let k be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, for some prime number p. Then, as is well-known from *local class field theory*, we have a *natural isomorphism*

$$(k^\times)^\wedge \xrightarrow{\sim} G_k^{\mathrm{ab}}$$

[where the "$\wedge$" denotes profinite completion]. By applying this isomorphism to the various open subgroups of G_k, we conclude that G_k is ab-*faithful*. On the other hand, it follows from the existence of nontrivial roots of unity in k that G_k *fails to be* ab-*torsion-free*, despite that fact that it is *torsion-free* [cf. [NSW], Corollary 12.1.3; [NSW], Theorem 12.1.7].

Example 6. Surface and Configuration Space Groups.

(i) Let G be a *pro-*Σ *surface group* [cf. [Mzk2], Definition 1.2]. Then, as is well-known, G is ab-*torsion-free* [cf. [Mzk2], Remark 1.2.2]. Moreover, one may verify easily that G is ab-*faithful*. [Indeed, since we are only concerned with the profinite group G up to isomorphism, we may assume without loss of generality that G arises from a hyperbolic curve of genus ≥ 1. Now let $N \subseteq H$ be subgroups as in Definition 1, (ii). If N corresponds to a hyperbolic curve of genus ≥ 2, then the desired injectivity follows as in the proof of [Mzk2], Proposition 1.4. If N corresponds to a hyperbolic curve of genus 1 [so $N \subseteq H$ corresponds to an isogeny of elliptic curves], then the desired injectivity follows immediately from an easy explicit computation of N^{ab}, H^{ab}.]

(ii) Let G be a *pro-*Σ *configuration space group* [cf. [Mzk2], Definition 2.3, (i)], where Σ is either equal to the set of all prime numbers or of cardinality one, that arises from a configuration space of dimension ≥ 2 associated to a hyperbolic curve of genus ≥ 2. Then it follows immediately from [Mzk2], Theorem 4.7, that G *fails to be* ab-*torsion-free*. On the other hand, it is not clear to the author at the time of writing whether or not G is ab-faithful.

Example 7. Absolute Galois Groups of Function Fields. [This example was related to the author by A. Tamagawa.] Let k be an *algebraically closed field of characteristic zero*; V a normal proper variety over k; K the function field of V; G_K the absolute Galois group of K; G the *maximal pro-*Σ *quotient* of G_K. Write $\mathrm{Div}(V)$ for the free abelian group generated by the irreducible divisors of V. Then:

(i) By assigning to an element of $K^\times$ its associated divisor of zeroes and poles, we obtain a homomorphism

$$K^\times \to \mathrm{Div}(V)$$

— which, as is well-known, determines an *injection* $\mathbb{P}(K) \stackrel{\mathrm{def}}{=} K^\times/k^\times \hookrightarrow \mathrm{Div}(V)$. In particular, it follows that $\mathbb{P}(K)$ is a *free abelian group*, so the natural homomorphism

$$\mathbb{P}(K) \to \varprojlim_N \ \mathbb{P}(K) \otimes \mathbb{Z}/N\mathbb{Z}$$

— where N ranges, in a multiplicative fashion, over the Σ-integers — is an *injection*.

(ii) Note that any k-linear automorphism α of K that induces the identity automorphism on $\mathbb{P}(K)$ is itself the identity. [Indeed, this follows from the observation that if $\alpha(f) = \lambda \cdot f$, $\alpha(f+1) = \mu \cdot (f+1)$, for $\lambda, \mu \in k^\times$ and $f \in K^\times$ such that $f \notin k^\times$, then [as is easily verified] $\lambda = \mu = 1$.]

(iii) Since k is algebraically closed, it follows immediately from *Kummer theory* [applied to K] that we have a *natural isomorphism*

$$\mathbb{P}(K) \otimes \mathbb{Z}/N\mathbb{Z} \stackrel{\sim}{\to} \mathrm{Hom}(G, \mathbb{Z}/N\mathbb{Z}(1)) \ (\cong \mathrm{Hom}(G, \mathbb{Z}/N\mathbb{Z}))$$

for any Σ-integer N. From this isomorphism [applied to the various open subgroups of G], one concludes immediately, in light of the observations of (i) and (ii), that G is ab-*torsion-free* and ab-*faithful*.

Bibliography

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